CALCULUS OF VARIATIONS AND APPLICATIONS
A CONFERENCE TO CELEBRATE
GIANNI DAL MASO'S 65TH BIRTHDAY

JANUARY 27 - FEBRUARY 1, 2020
SISSA, TRIESTE

PROGRAMME

Monday January 27

9:30 Registration
9:45 Opening
10:00 **Ambrosio**
11:00 coffee break
11:30 **Fonseca**
12:30 lunch break
15:00 **Chambolle**
16:00 coffee break
16:30 **Leoni**
18:00 *SISSA Colloquium:* **Barish**

Tuesday January 28

9:30 **Ambrosetti** (skype call)
9:45 **Mielke**
10:45 coffee break
11:15 **Sbordone**
12:15 lunch break
14:15 **Conti**
15:15 **Marcellini**
16:15 coffee break
16:45 **Braides**
20:00 social dinner

Wednesday January 29

9:30 **Fusco**
10:30 coffee break
11:00 **Savaré**
12:00 lunch break
14:00 **De Philippis**
15:00 **Alberti**
16:00 coffee break
16:30 **De Simone**

Thursday January 30

9:30 **Boccardo**
10:30 coffee break
11:00 **Dell'Antonio**
12:00 lunch break
14:00 **Francfort**
15:00 **Bouchitté**
16:00 coffee break
16:30 **Buttazzo**

Friday January 31

9:30 **Frankowska**
10:30 coffee break
11:00 **Morel**
12:00 lunch break
14:00 **Larsen**
15:00 coffee break
15:30 **Murat**

Saturday February 1

Informal discussions

ABSTRACTS

Minimal planar N-partitions for large N

Giovanni Alberti
University of Pisa

A minimal N-partition of a given planar domain E is a partition which consists of N sets (cells) with equal area, that minimize the total perimeter (that is, the length of the union of the boundaries of all cells). T. C. Hales proved in 2001 that if E is a flat 2-dimensional torus then the regular hexagonal partition (when it exists) is the unique minimal N-partition.

It is then interesting to understand what happens for a planar domain E that does not admit a regular hexagonal N-partition; in particular the following questions naturally arise: are the cells asymptotically hexagonal as N tends to infinity, and to which extent the partition looks locally hexagonal? Is the partition rigid, in the sense that the orientation of the cells is (essentially) the same through the domain? Similar questions arise also in the study of the asymptotic shape of minimal N-clusters.

In this talk I will describe some results obtained in these directions together with Marco Caroccia (Scuola Normale and University of Firenze) and Giacomo Del Nin (University of Warwick).

Semigroups and Geometric Measure Theory

Luigi Ambrosio
Scuola Normale Superiore, Pisa

I will illustrate with a few examples how semigroup tools can have a crucial role in the proof extensions of classical results in Geometric Measure Theory and Real Analysis.

Linear (and nonlinear) Dirichlet problems with singular convection/drift term

Luccio Boccardo
Sapienza University, Roma

We discuss the existence of distributional solutions for the boundary value problems (the first with a convection term, the second with a drift term)

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(uE(x)) + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

$$\begin{cases} -\mathrm{div}(M(x)\nabla \psi) = E(x)\nabla\psi + g(x) & \text{in } \Omega\,, \\ \psi = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We note that, at least formally, the two above linear problems are in duality and that the differential operators may be not coercive, unless one assumes that either the norm of $|E|$ in $L^N(\Omega)$ is small, or that $\mathrm{div}(E) = 0$.

We assume that Ω is a bounded open subset of $\mathbb{R}^N$,

$$E \in (L^N(\Omega))^N;\ f,\, g \in L^m(\Omega),\ m \geq 1,$$

$M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix such that (for $\alpha,\, \beta \in \mathbb{R}^+$)

$$\alpha|\xi|^2 \leq M(x)\xi\,\xi \qquad |M(x)| \leq \beta \quad \text{a.e. in } \Omega\,,\ \forall \xi \in \mathbb{R}^N$$

and we prove the Stampacchia-Calderon-Zygmund results of the case $E = 0$.
If $E \notin (L^N(\Omega))^N$, even for "nothing", as in $|E| \leq \frac{|A|}{|x}$, $A \in \mathbb{R}$, or if we add, in the boundary value problems, the zero order term "+u", the framework changes completely.

A repulsive multi-marginal transport model in quantum chemistry

Guy Bouchitté
University of Toulon

We propose a duality theory for multi-marginal repulsive cost that appear in optimal transport problems and in Density Functional Theory (quantum chemistry) The related optimization problems involve probabilities on the entire space and, as minimizing sequences may lose mass at infinity, it is natural to expect relaxed solutions which are sub-probabilities. We first characterize the N-marginals relaxed cost in terms of a stratification formula which takes into account all k interactions with $k \leq N$. We then develop a duality framework involving continuous functions vanishing at infinity and deduce primal-dual necessary and sufficient optimality conditions Next we prove the existence and the regularity of an optimal dual potential under very mild assumptions. In a last part of the talk, we apply our results to a minimization problem involving a given continuous potential and we give evidence of a mass quantization effect for optimal sub-probabilities.

This is a joint work with G. Buttazzo, T.Champion, and L. De Pascale.

Homogenization of ferromagnetic energies on Poisson random sets in the plane

Andrea Braides
University of Roma "Tor Vergata"

We prove that by scaling nearest-neighbour ferromagnetic energies defined on Poisson random sets in the plane we obtain an isotropic perimeter energy with a surface tension characterised by an asymptotic formula. The result relies on proving that cells with 'very long' or 'very short' edges of the corresponding Voronoi tessellation can be neglected and on computing effective metric on some random sets. Work in collaboration with A.Piatnitski.

Relations between torsional rigidity and principal eigenvalue

Giuseppe Buttazzo
University of Pisa

The relations between the principal eigenvalue of the Laplace operator in a domain Ω, with Dirichlet boundary conditions, and torsional rigidity, that is $\int_\Omega u\,dx$, where u solves $-\Delta u = 1$ in Ω with zero trace on the boundary, are studied in the class of general domains, convex domains, and domains with a small thickness. This is of help to provide some bounds for the Blaschke-Santaló diagram of the two quantities. This is an ongoing research with Michiel van den Berg (Bristol) and Aldo Pratelli (Pisa).

Some results on Dal Maso's GSBD functions

Antonin Chambolle
École Polytechnique, Paris

I will review in this talks a rigidity and a few approximation results obtained in the past years with S. Conti, V. Crismale, G. Francfort, F. Iurlano, and this year together with F. Cagnetti and L. Scardia, on the class "GSBD" introduced in 2011 by Gianni Dal Maso as the natural energy space for the weak variational formulation of fracture growth problems.

A phase-space formulation of the theory of elasticity and its relaxation

Sergio Conti
University of Bonn

We study a phase-space formulation of infinitesimal and finite elasticity, based on the data-driven approach by Kirchdoerfer and Ortiz (CMAME 2016). We present a framework for obtaining existence with the direct method, based on suitable concepts of coercivity and closedness of the data set, and provide examples of frame-indifferent data sets with those properties. For linear elasticity we also present a framework for the theory relaxation, which turns out to be fundamentally different from the classical relaxation of energy functions. For instance, we show that in the Data-Driven framework the relaxation of a bistable material leads to material data sets that are not graphs. This talk is based on joint work with Stefan Müller and Michael Ortiz.

Regularity of the free boundary for the two-phase Bernoulli problem

Guido De Philippis
SISSA, Trieste and Courant Institute, New York

I will illustrate a recent result obtained in collaboration with B. Velichkov and L. Spolaor concerning the regularity of the free boundaries in the two phase Bernoulli problems. The new point is the analysis of the free boundary close to branch points, where we show that it is given by the union of two C^1 graphs. This completes the analysis started by Alt, Caffarelli, and Friedman in the 80's.

Shape programming of active surfaces

Antonio De Simone
SISSA, Trieste and Scuola Superiore Sant'Anna, Pisa

We will discuss some recent results on biological and bio-inspired morphing, and use them to identify promising research directions for the future. In particular, we consider issues related to morphing at microscopic scales inspired by unicellular organisms. In this context of cell envelopes, active surfaces are thin layers of active gel materials made of assemblies of filaments and motors. These structures can be conveniently modeled as liquid crystal active films. Conversely, these structures may be used to inspire new morphing strategies for thin films made of liquid crystal elastomers, or of more general active materials. We will focus on general conceptual principles of broad applicability and, in particular, on morphing approaches based on the use of Gauss'theorema egregium (Gaussian morphing).

Contact interactions in Quantum Mechanics and Gamma convergence

Gianfausto Dell'Antonio

Sapienza University, Roma and SISSA, Trieste

We shall discuss contact interactions in Quantum Mechanics, as representatives of interactions of extremely short range. They have a role, e.g., in low energy physics (Efimov effect, i.e. presence of infinitely many bound states) and in the formulation of the Gross-Pitaievskii equation for the Bose-Einstein condensate. The mathematics is self-adjoint extensions of symmetric operators (or of quadratic forms) but the strategy takes inspiration from the theory of composite materials and makes essential use, inter alia, of Gamma convergence.

A Homogenization Result in the Gradient Theory of Phase Transitions

Irene Fonseca

Carnegie Mellon University, Pittsburgh

A variational model in the context of the gradient theory for fluid-fluid phase transitions with small scale heterogeneities is studied. Several regimes are considered depending on the ratio between this scale and that governing the energy barrier of the phase transition.

Hyperbolicity in Von-Mises plasticity

Gilles Francfort

University Paris 13 and Courant Institute, New York

In this joint work with J.F. Babadjian, inspired from prior work with A. Giacomini and J.J. Marigo, we start an investigation of spatial hyperbolicity in Von Mises elasto-plasticity, the ultimate goal being an adjudication of the uniqueness of the plastic strain. After discussing a specific example where uniqueness, or lack thereof, can be established, I will present partial results, focussing on a 2d simplified model.

Abnormal Hamilton-Jacobi Equations and Abnormal Necessary Optimality Conditions arising in Infinite Horizon Problems

Hélène Frankowska

CNRS and Sorbonne Université, Paris

This talk is devoted to an infinite horizon optimal control problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x,u) of the control system

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \geq t_0 \\ x(t_0) = x_0, \end{cases}$$

that may be also subject to a state constraint. Its history goes back to F.P. Ramsey, 1928. The value function V may be discontinuous and may admit infinite values.

Infinite horizon problems exhibit many phenomena not arising in the context of finite horizon ones. Among such phenomena let us recall that already in 1970ies it was observed that in the necessary optimality conditions it may happen that the co-state is different from zero at infinity and that only abnormal conditions hold true (even for problems without state constraints). A way to avoid it is to impose assumptions guaranteeing the local Lipschitz continuity of $V(t,\cdot)$ with the Lipschitz constant becoming zero at infinity. Then $V(t,\cdot)$ has locally bounded superdifferentials and this is crucial for getting normal necessary optimality conditions. However, such regularity of the value function requests very strong assumptions on the Lagrangian L, much stronger that in the case of finite horizon problems. In general, superdifferentials are not bounded and may become "horizontal" leading to abnormal optimality conditions.

The value function solves a Hamilton-Jacobi equation in a generalized sense, that may, in turn, involve an abnormal Hamiltonian because of these horizontal sub/superdifferentials.

Despite these facts both necessary optimality conditions and uniqueness of solutions to the Hamilton-Jacobi equation hold true with such extended notions of super/subdifferentials.

References

[1] Basco V. & Frankowska H. (2019) *Hamilton-Jacobi-Bellman equations with time-measurable data and infinite horizon*, Nonlinear Differ. Equ. Appl., 26:7 https://doi.org/10.1007/s00030-019-0553-y

[2] Cannarsa P. & Frankowska H. (2018) *Value function, relaxation, and transversality conditions in infinite horizon optimal control*, J. Math. Anal. Appl., 457, 1188–1217

[3] Dal Maso G. & Frankowska H. (2000) *Uniqueness of solutions to Hamilton-Jacobi equations arising in calculus of variations*, in Optimal Control and Partial Differential Equations, J.Menaldi, E.Rofman and A.Sulem Eds., IOS Press, 335-345

[4] Dal Maso G. & Frankowska H. (2000) *Value function for Bolza problem with discontinuous Lagrangian and Hamilton-Jacobi inequalities*, ESAIM-COCV, 5, 369-394

[5] Dal Maso G. & Frankowska H. (2003) *Autonomous integral functionals with discontinuous nonconvex integrands : Lipschitz regularity of minimizers, Du Bois-Reymond necessary conditions and Hamilton-Jacobi equations*, Applied Mathematics and Optimization, 48, 39-66

[6] Frankowska H. (2020) *Infinite horizon optimal control for non-convex problems under state constraints*, Advances in Mathematical Economics, 23, Editor: T. Maruyama, https://www.springer.com/gp/book/9789811507120

Stability of the Riesz Potential Inequality

Nicola Fusco

University "Federico II", Napoli

Given a measurable set E we consider the Riesz Potential

$$\mathcal{F}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\alpha}}\, dxdy$$

where $0 < \alpha < n$.

It is well known that the maximum of $\mathcal{F}(E)$ among all sets of the same volume of E is achieved at a ball B, i.e.

$$\mathcal{F}(E) \leq \mathcal{F}(B) \qquad |E| = |B|.$$

Moreover, equality holds if and only if E is a ball. We shall discuss the stability of this inequality.

This is a joint work with Aldo Pratelli.

Variational fracture with Neumann boundary conditions

Chris Larsen
Worcester Polytechnic Institute

The usual variational formulation for Neumann boundary conditions with fracture is ill-posed, at least with global minimization. We will describe a different formulation (with global minimization) that allows existence.

On higher-order Gamma-convergence

Giovanni Leoni
Carnegie Mellon University, Pittsburgh

In this talk, we will discuss some recent applications of asymptotic development by Gamma-convergence to singularly perturbed problems.

Regularity for elliptic equations and systems under either slow or fast growth conditions

Paolo Marcellini
University of Firenze, Italy

We obtain an *a-priori* $W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)-$bound for weak solutions to the elliptic system

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha (x, Du) = 0, \qquad \alpha = 1, 2, \ldots, m,$$

where Ω is an open set of $\mathbb{R}^n$, $n \geq 2$, u is a vector-valued map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. The vector field $A(\xi) = (a_i^\alpha(\xi))_{i=1,2,\ldots,n}^{\alpha=1,2,\ldots,m}$, $A : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$, of class $C^1(\mathbb{R}^{m\times n})$, has a variational nature in the sense that $a_i^\alpha(x,\xi) = \frac{\partial f}{\partial \xi_i^\alpha} = f_{\xi_i^\alpha}$ for every $i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$, where $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a convex *Carathéodory integrand*; i.e., $f = f(x,\xi)$ is measurable with respect to $x \in \mathbb{R}^n$ and it is a convex function with respect to $\xi \in \mathbb{R}^{m\times n}$. In this context of vector-valued maps and systems a classical assumption finalized to the *everywhere regularity* is a modulus-dependence in the energy integrand; i.e., we require that

$$f(x,\xi) = g(x, |\xi|)\ ,$$

where $g(x,t)$, $g : \Omega \times [0,\infty) \to [0,\infty)$, is a Carathéodory function, convex and increasing with respect to the gradient variable $t \in [0,\infty)$. We allow $x-$dependence, which turns out to be a relevant difference with respect to the autonomous case and not only a technical perturbation. Our assumptions allow us to consider *both fast and slow growth.* We consider fast growth even of *exponential type*; and slow growth, for instance of *Orlicz-type* with energy-integrands such as $g(x, |Du|) = |Du| \log(1+|Du|)$ or, when $n = 2, 3$, even *asymptotic linear growth* with energy integrals of the type

$$\int_\Omega g(x, |Du|)\, dx = \int_\Omega \left\{ |Du| - a(x)\sqrt{|Du|} \right\} dx\,.$$

The most recent results described here are obtained in joint work (submitted, under referee) with Tommaso Di Marco *(University of Firenze, Italy).*

Finite-strain viscoelasticity with temperature coupling

Alexander Mielke

Weierstraß Institut für Angewandte Analysis und Stochastik and
Humboldt Universität zu Berlin

We describe recent work with Tomas Roubíček concerning the coupling of finite-strain viscoelasticity with temperature effects. Major nonlinearities arise through the static and time-dependent frame indifference, the thermomechanical coupling, and the viscous heating. To tackle the difficulties, we consider a regularized model (a so-called second-grade material) such that the deformation tensor is continuous and has a determinant bounded away from 0. The latter relies on a uniform version of the Healey-Krömer estimate. To control the time derivatives we rely on a generalized Korn inequality developed by Neff and Pompe.

A. Mielke, T. Roubíček. Thermoviscoelasticity in Kelvin-Voigt rheology at large strains. WIAS preprint 2584, arXiv:1903.11094.

Image segmentation: from the Mumford-Shah conjecture to more global variational formulations by deep learning

Jean-Michel Morel

École Normale Supérieure, Cachan

In this talk I'll first recall to the (still unsolved!) image segmentation problem. I'll review briefly the more and more accurate results that have been obtained toward the conjecture, which is still open, to the best of my knowledge. Then I'll enlarge the topic and return to the fundamental problem: what is a segmentation? I'll establish the link with current variational methods used in machine learning to detect cracks in images.

Homogenization of the Neumann's brush problem

François Murat

Laboratoire Jacques-Louis Lions, Sorbonne Université and CNRS, Paris

In this lecture, I will describe joint work with Antonio Gaudiello (Naples, Italy) and Olivier Guibé (Rouen, France).

We consider a sequence of domains Ω^ε which have the form of brushes (in dimension $N=3$) or of combs (in dimension $N=2$). Each domain Ω^ε is an open subset of $\mathbf{R}^N$ with $N \geq 2$ which is made of teeth distributed over a basis. When ε tends to zero, the basis and the height of the teeth remain fixed, while the teeth vary in diameters, forms and distributions. For each domain Ω^ε, each tooth is assumed to be a vertical cylinder with a fixed height independent of ε and a diameter which is less than or equal to ε. The cross sections of the teeth, which are open, can vary from one tooth to another one; they are not assumed to be smooth; moreover the teeth can be adjacent, i.e. they can share parts of their boundaries. No periodicity is assumed on the distributions of the teeth. Finally the sequence of the characteristic functions of the cross sections of the teeth is assumed to have, as ε tends to zero, an $L^\infty(\mathbf{R}^{N-1})$ weak-star limit $\theta = \theta(x')$ (this latest assumption is an innocuous one).

For this sequence of domains we study the asymptotic behavior, as ε tends to zero, of the solution of a linear second order elliptic equation with a zeroth order term which is bounded from below away from zero, and with a source term in L^2, when the homogeneous Neumann boundary condition is imposed on the boundaries of the domains Ω^ε.

This is a classical homogenization problem since the pioneering work presented by R. Brizzi and J.-P. Chalot in their Ph.D. Thesis in 1978, but our work takes place in a geometry which is much more general that the ones which have been considered since that time. In our paper A. Gaudiello, O. Guibé & F. Murat, Homogenization of the brush problem with a source term in L^1, published in Archive for Rational Mechanics and Analysis, volume 225, (2017), pp. 1-64, we have treated the case where $\theta_0 \leq \theta(x') \leq 1$ for some $\theta_0 > 0$ and given for the first time a corrector result. I will state and prove these homogenization and corrector results.

I will also describe a very recent work in which we treat the case where $\theta_0 = 0$. In this case the limit problem is in general a degenerated one in the zone of the teeth, since in this zone it can exist a vertical measurable cylinder where no matter remains at the limit.

Singular perturbation of gradient flows and rate-independent evolution

Giuseppe Savaré

University of Pavia

We will present some new results concerning limits of singularly perturbed gradient flows of non-convex functionals in infinte dimensional Hilbert spaces. A particularly important case arises when the total variation of the approximating curves is not uniformly bounded and one has to recover a limit by combining Kuratowski convergence of the graphs with a variational description of the energy dissipation. This approach leads to a new notion of dissipative solution for an evolving family of stationary problems parametrized by the time variable. (In collaboration with Virginia Agostiniani and Riccarda Rossi)

Atomic decompositions and two stars theorems for non reflexive Banach function spaces

Carlo Sbordone

University "Federico II", Napoli

The o-O theory of non reflexive Banach spaces deals with a "large space" E defined by a big-O condition and a "small space" $E_0 \subset E$ given by the corresponding little-o condition. Spaces of this kind include $BMO(Q_0)$, $BV(Q_0)$ and $Lip_\alpha(Q_0)$, $0 < \alpha \leq 1$ on the unit cube $Q_0 \subset \mathbb{R}^n$.

Most of these spaces have a predual E_* with atomic decomposition. Another typical result, when the "vanishing" space E_0 is sufficiently rich, is the "two stars" theorem

$$E_* \sim E_0^*$$

The new space $E = B(Q_0)$ of Bourgain-Brezis-Mironescu enjoys all these properties. On the contrary $E = BV(Q_0)$ and $E = Lip_1(Q_0)$ do not.

For (K, ρ) a compact metric space, $E = Lip(K, \rho)$ is dual of the normed space $\mathcal{M}(K, \rho)$ of Borel measures endowed with the Kantorovich norm. Moreover its completion $\mathcal{M}^c(K, \rho)$ satisfies the duality condition

$$\mathcal{M}^c(K, \rho) \cong E_0^* = lip_1(K, \rho)^*$$

if and only if a "density" condition holds true depending on the metric ρ.
(In cooperation with L. D'Onofrio - L. Greco - K.M. Perfekt - A. Popoli - R. Schiattarella.)